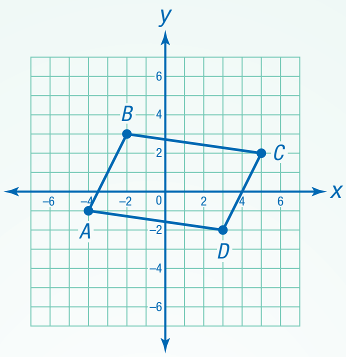
LESSON 23 Using Coordinate Geometry to Prove Theorems

EXAMPLE A Quadrilateral *ABCD* is shown on the coordinate plane.

Prove that *ABCD* is a parallelogram.



1 Make a plan.

A parallelogram is a quadrilateral in which opposite sides are parallel. To prove that *ABCD* is a parallelogram, find the slope of each side and show that the slopes of opposite sides are the same.

2 Find the slope of each side of the quadrilateral.

Use the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$\overline{AB}$: $m = \frac{3 - (-1)}{-2 - (-4)} = \frac{4}{2} = 2$

$\overline{BC}$: $m = \frac{2 - 3}{5 - (-2)} = \frac{-1}{7} = -\frac{1}{7}$

$\overline{CD}$: $m = \frac{-2 - 2}{3 - 5} = \frac{-4}{-2} = 2$

$\overline{DA}$: $m = \frac{-1 - (-2)}{-4 - 3} = \frac{1}{-7} = -\frac{1}{7}$

3 Analyze the results.

$\overline{AB}$ and $\overline{CD}$ are opposite sides, and they have the same slope, 2.

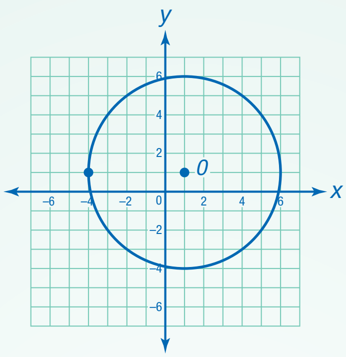
$\overline{BC}$ and $\overline{DA}$ are opposite sides, and they have the same slope, $-\frac{1}{7}$.

▶ Since both pairs of opposite sides are parallel, *ABCD* is a parallelogram.

DISCUSS

The definition of a rectangle is a parallelogram with four right angles. Is *ABCD* a rectangle? How do you know?

EXAMPLE B Circle O in the graph below has center $O(1, 1)$. The point $(-4, 1)$ lies on the circle. Prove that the point $(4, 5)$ also lies on circle O.



1 Make a plan.

The definition of a circle is all points that are equidistant from a given point, called the center. That distance is the radius. So, find the length of the radius, and then find the distance between $(4, 5)$ and the center. If those distances are equal, the point lies on the circle.

2 Find the length of the radius.

The point $(-4, 1)$ lies on the circle, so the radius is equal to the distance between $(-4, 1)$ and the center, $(1, 1)$. Since this segment is horizontal, find the difference of their x-coordinates to find the length.

$r = |1 - (-4)| = 5$

The radius of circle O is 5 units.

3 Find the distance between point O and $(4, 5)$. Compare it to the radius.

Find the distance between $(1, 1)$ and $(4, 5)$. Use the distance formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(5 - 1)^2 + (4 - 1)^2}$

$d = \sqrt{4^2 + 3^2}$

$d = \sqrt{25}$

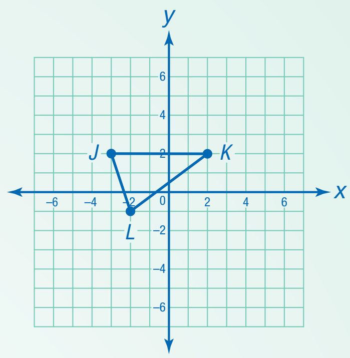
$d = 5$

▶ This is equal to the radius, so the point $(4, 5)$ does lie on circle O.

TRY

Circle C has center $(-5, -6)$ and radius $2\sqrt{3}$. Is the point $(-8, -4)$ on the circle?

EXAMPLE C Triangle *JKL* is shown on the coordinate plane on the right. Is $\triangle JKL$ a right triangle? Is it an isosceles triangle?.



1 Make a plan.

A right triangle contains one right angle. This means that two of the sides of the triangle will be perpendicular, so they will have slopes that are opposite reciprocals. Find the slopes of all sides of $\triangle JKL$.

An isosceles triangle has two congruent sides. Find the side lengths of $\triangle JKL$.

2 Find the slopes of all sides and compare them.

Use the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$\overline{JK}$: $m = \frac{2-2}{2-(-3)} = \frac{0}{5} = 0$

$\overline{KL}$: $m = \frac{2-(-1)}{2-(-2)} = \frac{3}{4}$

$\overline{JL}$: $m = \frac{2-(-1)}{-3-(-2)} = \frac{3}{-1} = -3$

▶ No two slopes are opposite reciprocals of each other, so $\triangle JKL$ is not a right triangle.

3 Find the lengths of the sides and compare them.

$\overline{JK}$ is horizontal, so find the difference of the *x*-coordinates to find its length.

$JK = |2 - (-3)| = 5$

For the remaining sides, use the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$KL = \sqrt{(-2-2)^2 + (-1-2)^2}$

$KL = \sqrt{25}$

$KL = 5$

$JL = \sqrt{(-2-(-3))^2 + (-1-2)^2}$

$JL = \sqrt{10}$

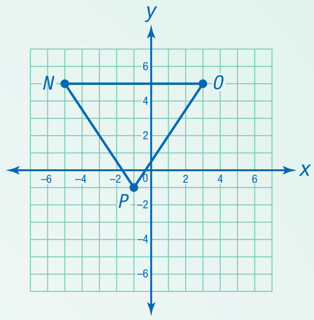
▶ $JK = KL = 5$, so $\triangle JKL$ is isosceles.

TRY

To what point could you move *L* to make *JKL* an isosceles right triangle?

EXAMPLE D Triangle *NOP* is shown on the coordinate plane on the right.

Prove that the centroid of $\triangle NOP$ divides each of the triangle's medians in a ratio of 2:1.



1 Find the endpoints of the medians of $\triangle NOP$.

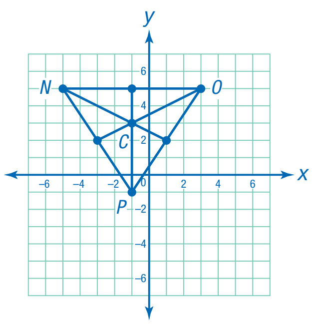
A median connects a vertex of a triangle to the midpoint of the opposite side. Find the midpoint of each side of the triangle. Use the formula $(x_1 + k(x_2 - x_1), y_1 + k(y_2 - y_1))$ with $k = \frac{1}{2}$.

$\overline{NO}$: $\left(-5 + \frac{1}{2}[3 - (-5)], 5 + \frac{1}{2}(5 - 5)\right) = (-1, 5)$

$\overline{OP}$: $\left(3 + \frac{1}{2}(-1 - 3), 5 + \frac{1}{2}(-1 - 5)\right) = (1, 2)$

$\overline{NP}$: $\left((-5 + \frac{1}{2}[-1 - (-5)], 5 + \frac{1}{2}(-1 - 5)\right) = (-3, 2)$

2 Graph the medians and find the centroid.



The centroid appears to be at $(-1, 3)$.

3 Find the points that partition each median in a ratio of 2:1.

A ratio of 2:1 indicates the point $\frac{2}{3}$ of the way from the vertex to the opposite side. Use the partition formula and let $k = \frac{2}{3}$.

Median from *N*:

$\left(-5 + \frac{2}{3}[1 - (-5)], 5 + \frac{2}{3}(2 - 5)\right) =$

$(-5 + 4, 5 + (-2)) = (-1, 3)$

Median from *O*:

$\left(3 + \frac{2}{3}(-3 - 3), 5 + \frac{2}{3}(2 - 5)\right) =$

$(3 + (-4), 5 + (-2)) = (-1, 3)$

Median from *P*:

$\left(-1 + \frac{2}{3}[-1 - (-1)], -1 + \frac{2}{3}[5 - (-1)]\right) =$

$(-1 + 0, -1 + 4) = (-1, 3)$

The point $(-1, 3)$ lies on all three lines, so it must be the centroid.

▶ The centroid, $(-1, 3)$, partitions each median in a ratio of 2:1.

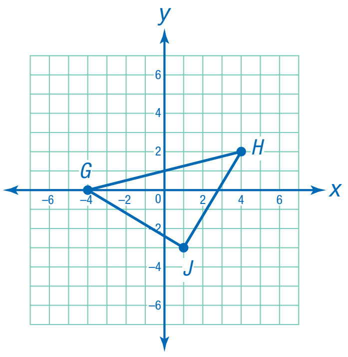
CHECK

Pick a median and find its length. Then, find the distance from the vertex of that median to $C(-1, 3)$. Is that distance $\frac{2}{3}$ the length of the median?

Practice

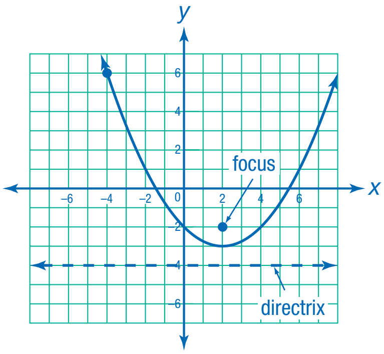
Solve.

1. Triangle *GHJ* is shown on the coordinate plane on the right. Is $\triangle GHJ$ a right triangle? Explain your answer.



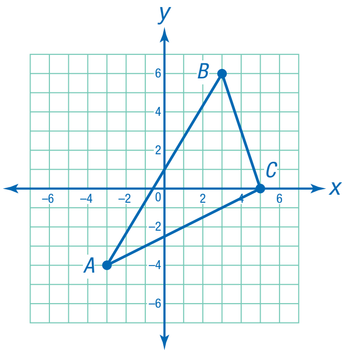
REMEMBER Lines that form a right angle are perpendicular to each other.

2. The point $(-4, 6)$ lies on the parabola graphed to the right. Prove that this point is equidistant from the focus, $(2, -2)$, and the directrix, $y = -4$.

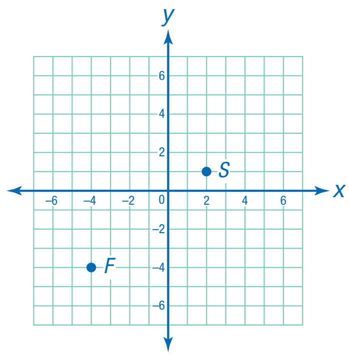


3. Triangle *ABC* is shown on the coordinate plane on the right.

Draw the line segment connecting the midpoint of $\overline{AB}$ to the midpoint of $\overline{BC}$ on the coordinate plane. Then prove that this line segment is parallel to $\overline{AC}$ and half its length.



4. The diagram on the right represents a park with a plane imposed on it. Each unit length on the plane represents 1 foot. The point *S* represents the planned placement for a sprinkler head that sprays water in a circle. The point *F* represents a flowerbed.



If the sprinkler has a radius of 6 feet, will the water from the sprinkler reach the flowerbed? Explain your answer.

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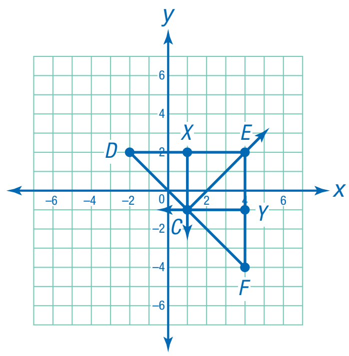
5. Right triangle *DEF* is shown on the coordinate plane on the right.



Find the intersection of the perpendicular bisectors of the triangle.

Prove that this point is the midpoint of $\overline{DF}$.

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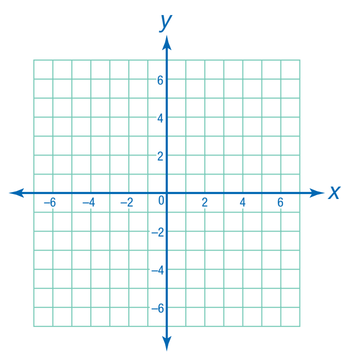
6. CONSTRUCT Draw a rhombus on the coordinate plane on the right so that no side of the rhombus is vertical or horizontal.



Prove that your figure is a rhombus. Then prove that your figure is also a parallelogram.

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